New Algorithmics for Polyhedral Calculus via Parametric Linear Programming

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Thèse réalisée à VERIMAG sous la direction de
David Monniaux et Michaël Périn

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13 Juin 2018

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// x ∈ [0,5]
// y ∈ [-5,5]
int z;
if (2*x + 8*y >= 11){
    z = 1/y;
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Is this feasible ?

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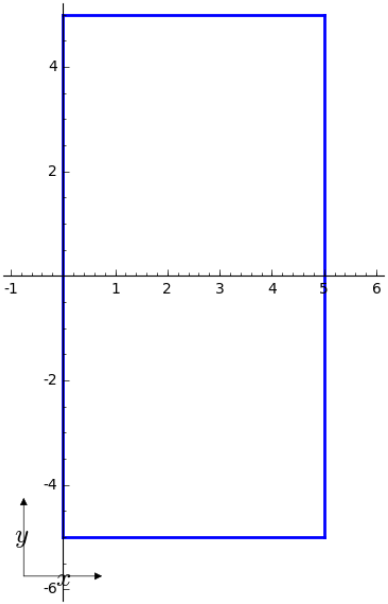

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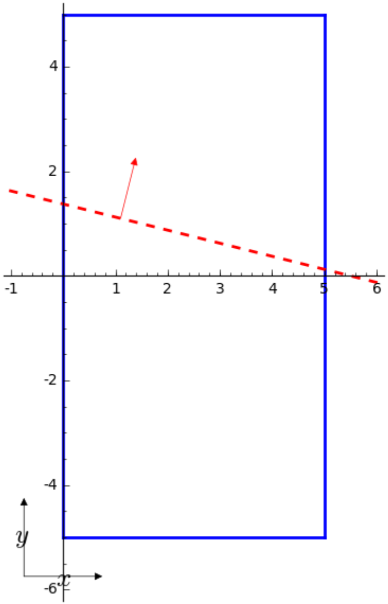

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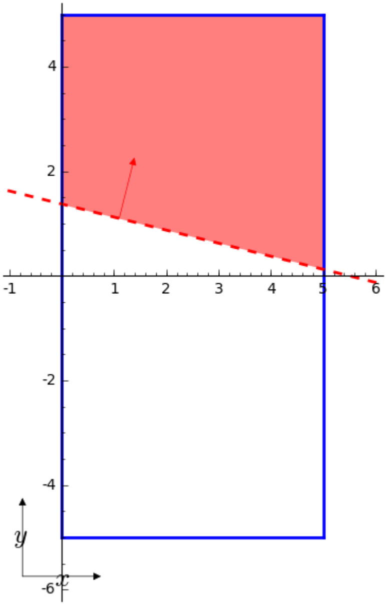
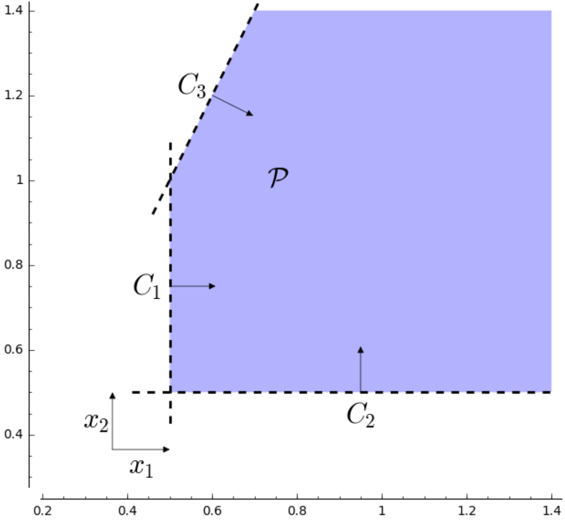


As **constraints**

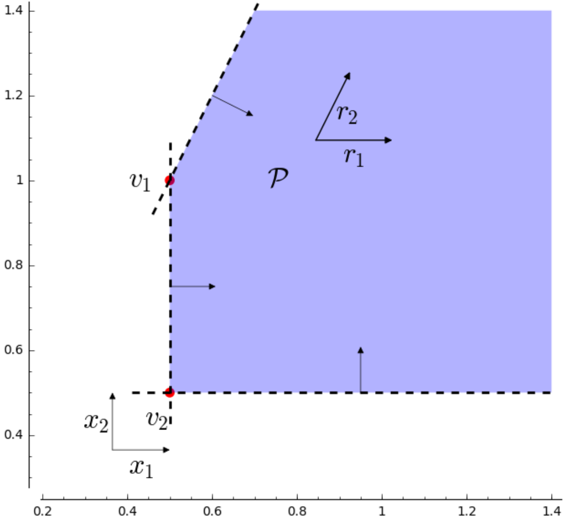


$\boldsymbol{C}_1 : 2x_1 \geq 1$

$\boldsymbol{C}_2 : 2x_2 \geq 1$

$\boldsymbol{C}_3 : 2x_1 - x_2 \geq 0$

As **generators**: vertices and rays



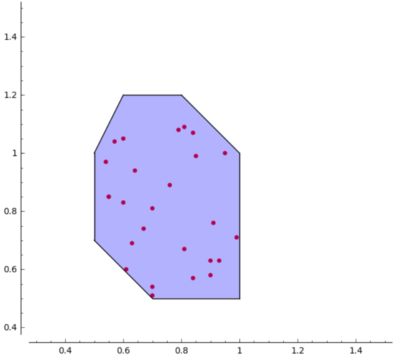
$\boldsymbol{v_1} : \left(\frac{1}{2}, 1\right)$

$\boldsymbol{v_2} : \left(\frac{1}{2}, \frac{1}{2}\right)$

$\boldsymbol{r}_1 : (1, 0)$

$\boldsymbol{r}_2 : (1, 1)$

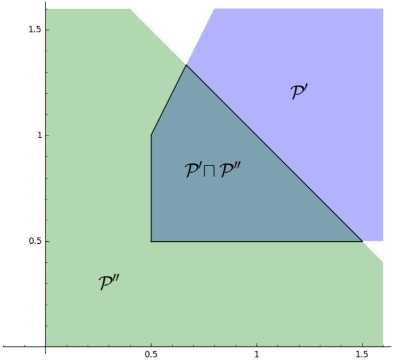
Abstract Interpretation: prove program properties by over-approximating reachable memory states



The Abstract domain of polyhedra [Cousot and Halbwachs, 1978]

- handles affine relations between variables;
- is powerful, but expensive

$$
\begin{array}{l}
\dots \\
/*\ \ (x_1, x_2) \in \mathcal{P}'\ \ */ \\
\texttt{if}\ \ (x_1 + x_2 \le 2)\{ \\
\qquad\qquad /*\ \ (x_1, x_2) \in \mathcal{P}' \sqcap \{x_1 + x_2 \le 2\}\ \ */ \\
\}
\end{array}
$$



$$
\mathcal{P}' : \begin{cases} \boldsymbol{C}_1 : 2x_1 \ge 1 \\ \boldsymbol{C}_2 : 2x_2 \ge 1 \\ \boldsymbol{C}_3 : 2x_1 - x_2 \ge 0 \end{cases}
$$

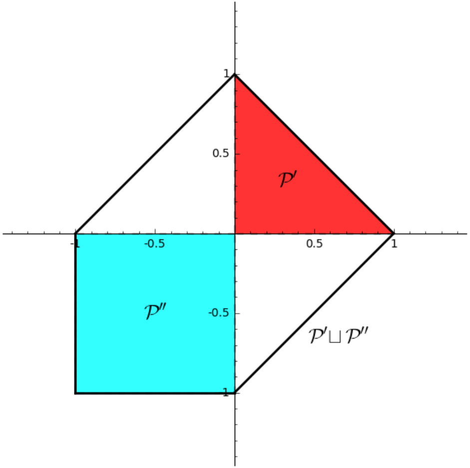
$$
\mathcal{P}'' : \left\{ \boldsymbol{C}' : x_1 + x_2 \le 2 \right.
$$

```
...
/*
if (...){
        ...
        (x1, x2) ∈ P'  */
}
else{
        ...
        (x1, x2) ∈ P''  */
}
/* (x1, x2) ∈ P' ⊔ P''  */
```

$(x_1, x_2) \in \mathcal{P}'$

$(x_1, x_2) \in \mathcal{P}''$

$(x_1, x_2) \in \mathcal{P}' \sqcup \mathcal{P}''$



Static Analyzers aim at verifying programs. For critical software

- false alarms are acceptable
- missed errors are not

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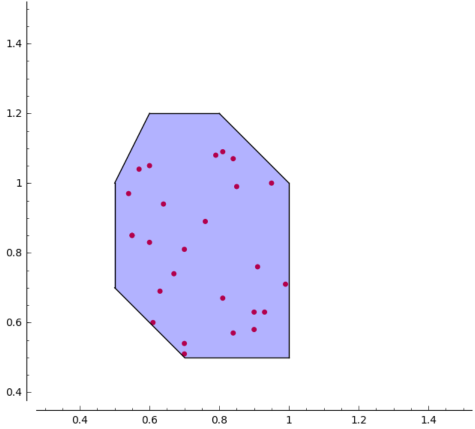
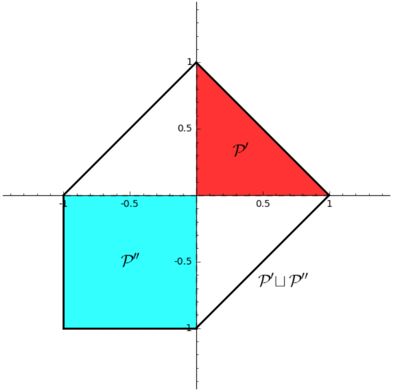
Thus, we wish to **certify the analyzer**.
It requires **certified operators**.

Correction of polyhedral operators

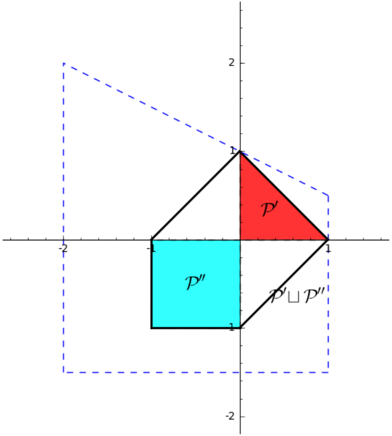
- In abstract interpretation, we over-approximate reachable memory states
- An operator is correct if its actual (computed) result includes the expected (mathematical) one
- Proofs are mainly inclusion





Correction of the convex hull $\mathcal{P}' \sqcup \mathcal{P}''$:

$$(\mathcal{P}' \sqsubseteq \mathcal{P}' \sqcup \mathcal{P}'') \wedge (\mathcal{P}'' \sqsubseteq \mathcal{P}' \sqcup \mathcal{P}'')$$



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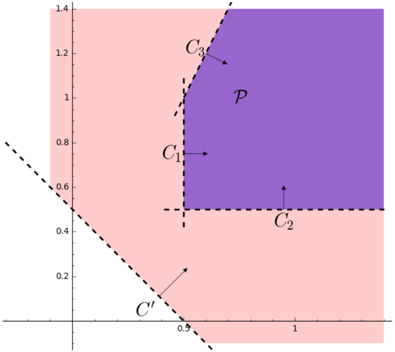

$$
\begin{array}{llcl}
\boldsymbol{C_1} & : & 2x_1 - 1 & \geq 0 \\
\boldsymbol{C_2} & : & x_2 - \frac{1}{2} & \geq 0 \\
\boldsymbol{C_3} & : & 2x_1 - x_2 & \geq 0 \\
 & & & \\
\boldsymbol{C'} & : & 2x_1 + x_2 - 1 & \geq 0
\end{array}
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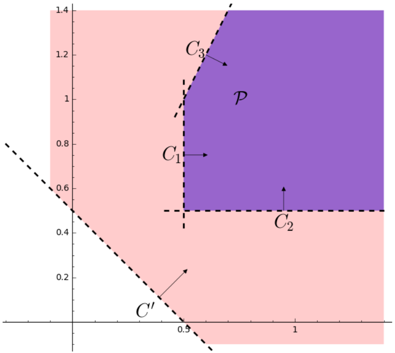

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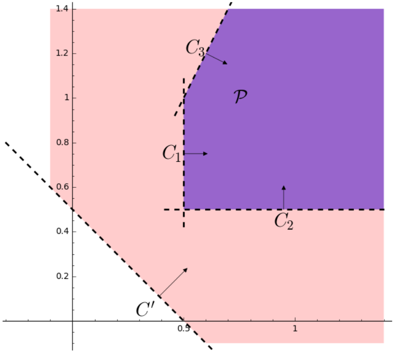
Farkas' combination: $\frac{1}{2} + 1 \cdot \boldsymbol{C_1} + 1 \cdot \boldsymbol{C_2}$

$$= \frac{1}{2} + 1 \cdot \underbrace{\left(2x_1 - 1\right)}_{\boldsymbol{C_1} \geq 0} + 1 \cdot \underbrace{\left(x_2 - \frac{1}{2}\right)}_{\boldsymbol{C_2} \geq 0} = \underbrace{2x_1 + x_2 - 1}_{\boldsymbol{C'}}$$



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The Verified Polyhedra Library (VPL) [Fouilhé et al., 2013]:

- Developed as a *relational* abstract domain in the certified analyzer Verasco [Jourdan et al., 2015]

- Polyhedral operators ($\sqsubseteq$, $\sqcup$, $\sqcap$, variable elimination) *a posteriori certified* in COQ
  - $\rightarrow$ results are computed by an OCAML program that generates Farkas' combinations
  - $\rightarrow$ simple verification of these combinations in COQ [Besson et al., 2010]

- *Constraint-only* representation of polyhedra

Double description

- Uses both representations of polyhedra
- Implemented by most polyhedra libraries
- Conversion from one representation to the other by Chernikova's algorithm

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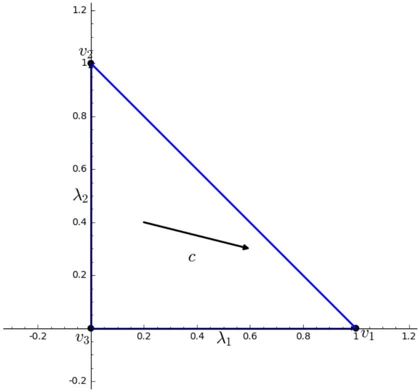
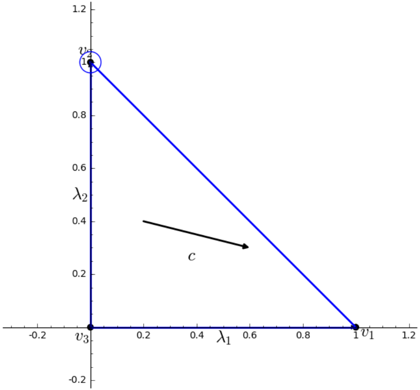
Idea: encode operators in Parametric Linear Programming



maximize *the objective* $\boldsymbol{c} \cdot (\lambda_1, \lambda_2) \triangleq 4\lambda_1 - \lambda_2$

under the constraints

$$\lambda_1 \geq 0$$
$$\lambda_2 \geq 0$$
$$\lambda_1 + \lambda_2 \leq 1$$



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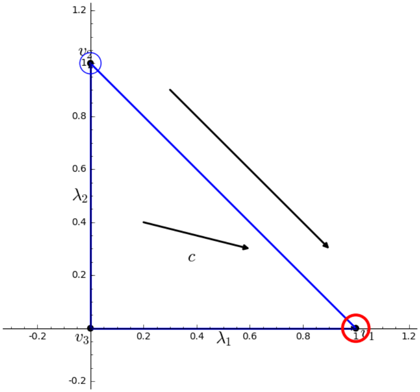

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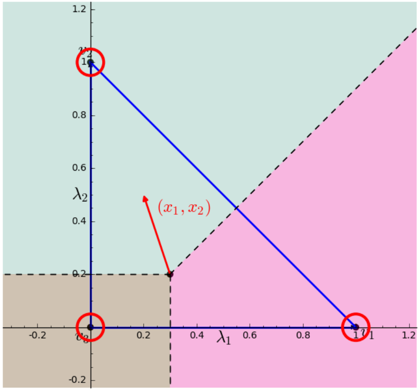

maximize *the objective* $\boldsymbol{c}(x_1, x_2) \cdot (\lambda_1, \lambda_2) \triangleq x_1\lambda_1 + x_2\lambda_2$

under the constraints

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$$\lambda_2 \geq 0$$
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Linear Programming	Parametric Linear Programming
minimize *the objective* z: $c_0 + c_1\lambda_1 + \ldots + c_m\lambda_m$ **under the constraints** $A\boldsymbol{\lambda} \leq b$	**minimize** *the objective* $z(\boldsymbol{x})$: $c_0(\boldsymbol{x}) + c_1(\boldsymbol{x}) \cdot \lambda_1 + \ldots + c_m(\boldsymbol{x}) \cdot \lambda_m$ **under the constraints** $A\boldsymbol{\lambda} \leq b$
$\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ are decision variables c_i's are **rational constants**	$\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ are decision variables c_i's are **affine forms of the parameters** $\boldsymbol{x} = (x_1, \ldots, x_n)$

PLP encoding of polyhedral operators

Let

$$op : polyhedra \longrightarrow polyhedra$$

be a polyhedral operator that must produce **only** over-approximations

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$$\forall \boldsymbol{x},\ \boldsymbol{C'}(\boldsymbol{x}) = \boxed{\lambda_0 + \lambda_1 \cdot \boldsymbol{C_1}(\boldsymbol{x}) + \ldots + \lambda_p \cdot \boldsymbol{C_p}(\boldsymbol{x})}$$

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Objective function of a PLP problem

PLP encoding of polyhedral operators

Generic form of the PLP encoding of a polyhedral operator:

$$\begin{gathered}\textbf{minimize } \lambda_0 + \lambda_1 \cdot \boldsymbol{C_1}(\boldsymbol{x}) + \ldots + \lambda_p \cdot \boldsymbol{C_p}(\boldsymbol{x}) \\ \textbf{under the constraints:} \\ A\boldsymbol{\lambda} \leq b \\ \boldsymbol{\lambda} \geq 0\end{gathered}$$

where

- $\boldsymbol{C_1}(\boldsymbol{x}) \geq 0, \ldots, \boldsymbol{C_p}(\boldsymbol{x}) \geq 0$ are the constraints of the input polyhedron $\mathcal{P}$
- $A\boldsymbol{\lambda} \leq b$ defines the operator

Conclusion

Parametric Linear Programming

- Generic tool (used for variable elimination, convex hull, linearization)
- Avoid redundancies in the result (thanks to a normalization constraint in the PLP encoding)

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Diffusion

- Available on Github (https://github.com/VERIMAG-Polyhedra)
- Opam package
- Frama-C binding: experimental branch of EVA

Perspectives

Ongoing work

- Use the VPL in Constraint Programming
- Binding in the AbSolute solver

Certification

- Provide precision certificates

Performance

- Cartesian product of polyhedra
- Parallelization of the PLP solver

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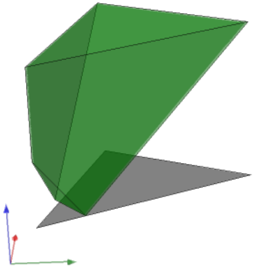

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\begin{array}{lcrrrcr}
\boldsymbol{C_1} & : & -x_1 & -2x_2 & +2x_3 & \geq & -7 \\
\boldsymbol{C_2} & : & -x_1 & +2x_2 & & \geq & 1 \\
\boldsymbol{C_3} & : & 3x_1 & -x_2 & & \geq & 0 \\
\boldsymbol{C_4} & : & & & -x_3 & \geq & -10 \\
\boldsymbol{C_5} & : & x_1 & +x_2 & +x_3 & \geq & 5
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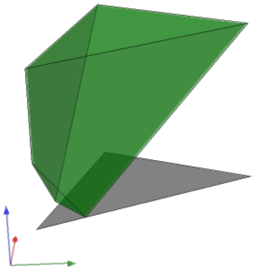
$$



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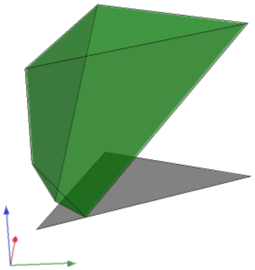
Fourier-Motzkin elimination

Kept constraints: $\boldsymbol{C_2}$; $\boldsymbol{C_3}$



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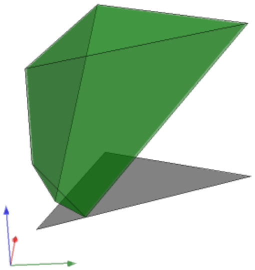
Combinations that cancel x_3: $\boldsymbol{C_1} + 2 \cdot \boldsymbol{C_4}$; $\boldsymbol{C_4} + \boldsymbol{C_5}$



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\boldsymbol{C_5} & : & x_1 & +x_2 & +x_3 & \geq & 5
\end{array}$$

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Combinations that cancel x_3: $\boldsymbol{C_1} + 2 \cdot \boldsymbol{C_4}$; $\boldsymbol{C_4} + \boldsymbol{C_5}$

Minimization returns irredundant solutions: $\{\boldsymbol{C_2}, \boldsymbol{C_3}, \boldsymbol{C_1} + 2 \cdot \boldsymbol{C_4}\}$

$$\boldsymbol{C_4} + \boldsymbol{C_5} = \frac{29}{5} + \frac{4}{5} \cdot \boldsymbol{C_2} + \frac{3}{5} \cdot \boldsymbol{C_3}$$

Projection is an important operator used for

- variable elimination
- convex hull

Computing projection by Fourier-Motzkin elimination:

$\rightarrow$ eliminates one variable at a time;

$\rightarrow$ generates many redundant constraints (the number of constraints can double at each elimination);

$\rightarrow$ has an exponential complexity in the number of eliminated variables

$\rightarrow$ "non-geometrical" algorithm: the order of elimination has a great impact on execution time

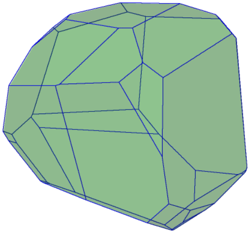
Projection problems randomly generated from:

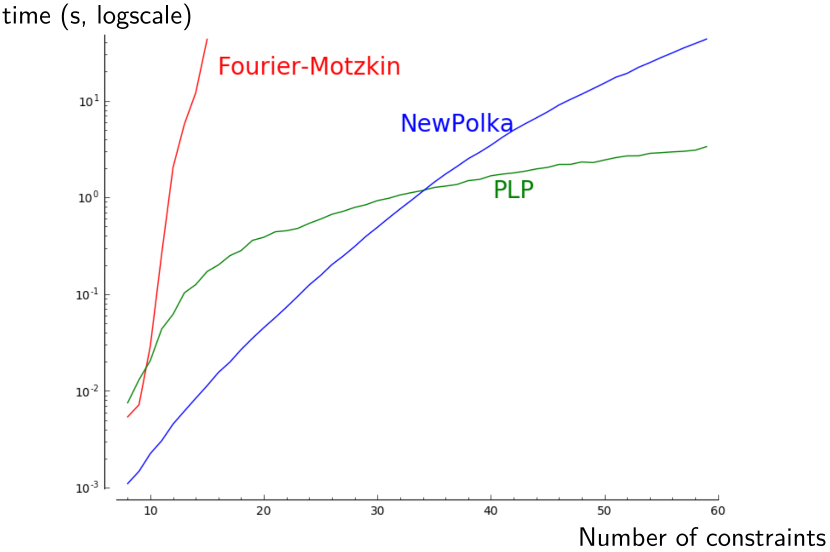
number of constraints

number of variables

density

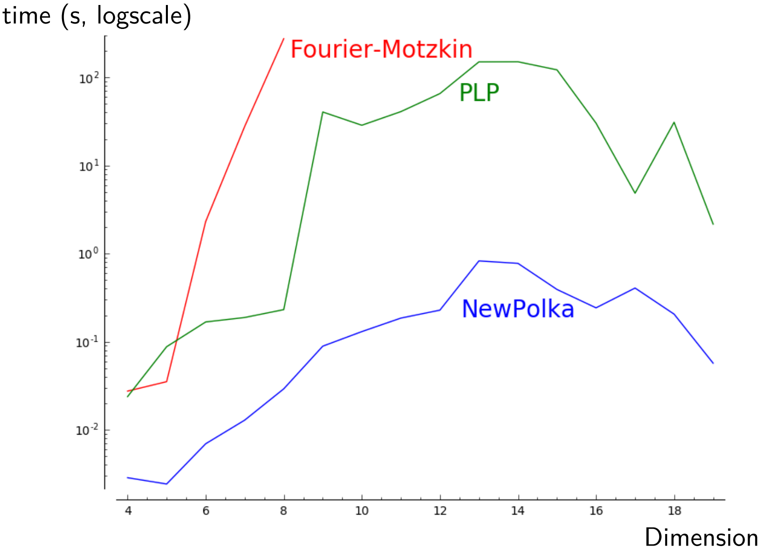
number of variables to eliminate





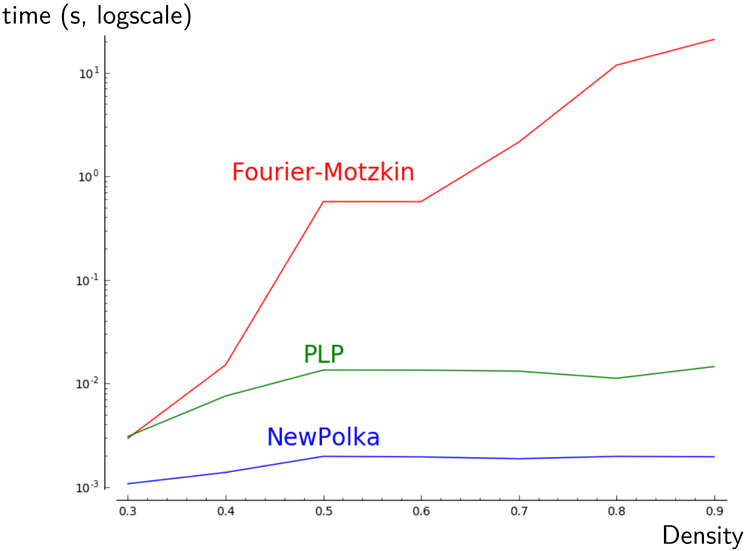
PLP scales better on big problems

Experiments on Projection via PLP



Fourier-Motzkin is still interesting for a small dimension

Experiments on Projection via PLP



PLP is less sensitive to density